

On the derivative at $t = 1$ of the skew-growth functions for Artin monoids.

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Abstract

Let G_M^+ be the Artin monoid of finite type generated by the letters $a_i, i \in I$ with respect to a Coxeter matrix M that is equipped with the degree map $\deg : G_M^+ \rightarrow \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words, and let $N_{M,\deg}(t) := \sum_{J \subset I} (-1)^{\#J} t^{\deg(\Delta_J)}$ be the skew-growth function, where the summation index J runs over all subsets of I and Δ_J is the fundamental element in G_M^+ associated to the set J . In this article, we will calculate the derivative at $t = 1$ of the polynomial $N_{M,\deg}(t)$. As a result, we show that the polynomial $N_{M,\deg}(t)$ has a simple root at $t = 1$.

Keywords: Artin monoid, growth function, zeroes of polynomial

1. Introduction

Let G_M^+ be the Artin monoid of finite type ([B-S]§1) generated by the letters $a_i, i \in I$ with respect to a Coxeter matrix M ([B]). Due to the homogeneity of the defining relations in G_M^+ , we naturally define a map $\deg : G_M^+ \rightarrow \mathbb{Z}_{\geq 0}$ defined by assigning to each equivalence class of words the length of the words. The *spherical growth function* for the monoid G_M^+ is defined as

$$P_{G_M^+, \deg}(t) := \sum_{u \in G_M^+} t^{\deg(u)}.$$

In [A-N][Bro][S1], they show that the inversion function $P_{G_M^+, \deg}(t)^{-1}$ is given by the following function, called the *skew-growth function*,

$$N_{M,\deg}(t) := \sum_{J \subset I} (-1)^{\#J} t^{\deg(\Delta_J)},$$

where the summation index J runs over all subsets of I and Δ_J is the fundamental element in G_M^+ associated to the set J ([B-S]§5). That has been investigated by several authors ([A-N][B][Bro][D][I1][I2][S1][S2][S3][S4][X]). In [B-S]§4, the authors show that the monoid G_M^+ satisfies the LCM condition (i.e. any two elements α and β in it admit the left (resp. right) least common multiple). By using this property, for a subset $J \subset I$, they defined the fundamental element Δ_J , as the right least common multiple of all the letters $a_i, i \in J$. In [S1]§4, it

is observed that the polynomial $N_{M,\deg}(t)$ has a simple root at $t = 1$. In this article, we will calculate the derivative at $t = 1$ of the polynomial $N_{M,\deg}(t)$. As a result, we show that the polynomial $N_{M,\deg}(t)$ has a simple root at $t = 1$.

Our main theorem is the following.

Theorem 1.1. For a Coxeter matrix M , the derivative at $t = 1$ of the polynomial $N_{M,\deg}(t)$ is given by the following list:

$$\begin{array}{ll} A_{l \geq 1} : & N'_{M,\deg}(1) = (-1)^l, & E_8 : & N'_{M,\deg}(1) = 44, \\ B_{l \geq 2} : & N'_{M,\deg}(1) = (-1)^l l, & F_4 : & N'_{M,\deg}(1) = 10, \\ D_{l \geq 4} : & N'_{M,\deg}(1) = (-1)^l (l - 2), & H_3 : & N'_{M,\deg}(1) = -8, \\ E_6 : & N'_{M,\deg}(1) = 7, & H_4 : & N'_{M,\deg}(1) = 42, \\ E_7 : & N'_{M,\deg}(1) = -16, & I_2(p \geq 5) : & N'_{M,\deg}(1) = p - 2. \end{array}$$

The above statement can be verified by hand calculation for the types $E_6, E_7, E_8, F_4, H_3, H_4$ and $I_2(p \geq 5)$. In §3, we will prove Theorem 1.1 for the type A_l . By using the results in §3, we will prove Theorem 1.1 for the type B_l and D_l in §4, §5. As a corollary of Theorem 1.1, we obtain the following.

Corollary 1.2. The polynomial $N_{M,\deg}(t)$ has a simple root at $t = 1$.

2. Preliminary results

Let l be a positive integer and let $I = \{1, 2, \dots, l\}$. The Coxeter matrix $M = (m(\alpha, \beta))_{\alpha, \beta \in I}$ of the type $X_l \in \{A_l, B_l, D_l\}$ is given by the following list. For the type A_l , we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 3 & \text{if } |\alpha - \beta| = 1 \\ 2 & \text{if } |\alpha - \beta| > 1 \end{cases}$$

For the type B_l ¹, we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 3 & \text{if } |\alpha - \beta| = 1 \text{ and } \alpha + \beta > 3 \\ 2 & \text{if } |\alpha - \beta| > 1 \\ 4 & \text{if } \alpha + \beta = 3 \end{cases}$$

For the type D_l , we give

$$m(\alpha, \beta) = \begin{cases} 1 & \text{if } \alpha = \beta \\ 2 & \text{if } |\alpha - \beta| > 1 \text{ and } \alpha + \beta \neq 2l - 2 \\ 2 & \text{if } \alpha + \beta = 2l - 1 \\ 3 & \text{if } |\alpha - \beta| = 1 \text{ and } \alpha + \beta < 2l - 1 \\ 3 & \text{if } \alpha + \beta = 2l - 2 \text{ and } \alpha \neq \beta \end{cases}$$

¹ For the type B_l , we adopt for convenience the different definition of the Coxeter matrix from that in [B].

We simply write the polynomial $N_{M,\deg}(t)$ by $N_{X_l}(t)$. Namely, we put

$$N_{X_l}(t) := \sum_{J \subset I} (-1)^{\#J} t^{\deg(\Delta_{X_l,J})},$$

where $\Delta_{X_l,J}$ is the fundamental element in the Artin monoid G_M^+ associated to the set J . Moreover, for a non-negative integer $j \in \{0, \dots, l\}$ we put

$$N_{X_l,j}(t) := \sum_{J \subset I, \#J=j} t^{\deg(\Delta_{X_l,J})}, \quad C_{X_l,j} := \left. \frac{dN_{X_l,j}(t)}{dt} \right|_{t=1}.$$

Therefore, we have the following equations:

$$N'_{X_l}(1) = \sum_{j=1}^l (-1)^j C_{X_l,j}, \quad C_{X_l,j} = \sum_{J \subset I, \#J=j} \deg(\Delta_{X_l,J}).$$

To a Coxeter matrix $M = (m(\alpha, \beta))_{\alpha, \beta \in I}$ of the type X_l , we attach a Coxeter graph Γ_{X_l} whose vertices are indexed by the set I and two vertices α and β are connected by an edge iff $m(\alpha, \beta) \geq 3$. For a subset $J \subset I$, we associate a full subgraph $\Gamma_{X_l}(J)$, whose vertices are indexed by the set J . The edge is labeled by $m(\alpha, \beta)$ (omitted if $m(\alpha, \beta) = 3$). We note that $\Gamma_{X_l}(I)$ corresponds to the graph Γ_{X_l} . For a subgraph $\Gamma_{X_l}(J)$ of Γ_{X_l} , we write the number of connected components of $\Gamma_{X_l}(J)$ by $k_{X_l}(J)$. Let $\Gamma_{X_l}(J)$ be a full subgraph of Γ_{X_l} with k -connected components $\Gamma_{X_l}(J_1), \Gamma_{X_l}(J_2), \dots, \Gamma_{X_l}(J_k)$. Then, we write

$$\Gamma_{X_l}(J) = \bigsqcup_{i=1}^k \Gamma_{X_l}(J_i).$$

We recall a fact from [B-S].

Proposition 2.1. For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{X_l}(J)$ has a decomposition $\Gamma_{X_l}(J) = \bigsqcup_{i=1}^k \Gamma_{X_l}(J_i)$. Then:

- (1) For $1 \leq i < j \leq k$, Δ_{X_l,J_i} and Δ_{X_l,J_j} commute with each other.
- (2) Then, the fundamental element $\Delta_{X_l,J}$ can be written as a product of the fundamental elements $\Delta_{X_l,J_1}, \dots, \Delta_{X_l,J_k}$:

$$\Delta_{X_l,J} = \Delta_{X_l,J_1} \cdots \Delta_{X_l,J_k}.$$

Since the map \deg is an additive map, we can compute

$$\deg(\Delta_{X_l,J}) = \sum_{i=1}^k \deg(\Delta_{X_l,J_i}). \quad (2.1)$$

3. Proof of the type A_l

Let l be a positive integer and let $I = \{1, 2, \dots, l\}$. In this section, we will prove Theorem 1.1 for the type A_l . First, we have a remark on $k_{A_l}(J)$.

Proposition 3.1. For a subset $J \subset I$, we put $j := \#J$. Then:

(1) If the number j satisfies an inequality $1 \leq j \leq \lceil \frac{l}{2} \rceil$, then the number of connected components $k_{A_l}(J)$ can run over from 1 to j .

(2) If the number j satisfies an inequality $j > \lceil \frac{l}{2} \rceil$, then the number of connected components $k_{A_l}(J)$ can run over from 1 to $l - j + 1$.

We put $\beta_{l,j} := \min\{j, l - j + 1\}$. Then, the summary of Proposition 3.1 is that the number of connected components $k_{A_l}(J)$ can run over from 1 to $\beta_{l,j}$. For two positive integers j and k with $j \leq l$ and $k \leq \beta_{l,j}$, we put

$$N_{A_l,j}^{(k)}(t) := \sum_{J \subset I, \#J=j, k_{A_l}(J)=k} t^{\deg(\Delta_{A_l,J})}, \quad C_{A_l,j}^{(k)} := \left. \frac{dN_{A_l,j}^{(k)}(t)}{dt} \right|_{t=1}.$$

Therefore, we have the following equation:

$$C_{A_l,j}^{(k)} = \sum_{J \subset I, \#J=j, k_{A_l}(J)=k} \deg(\Delta_{A_l,J}).$$

By definition, we have

$$C_{A_l,j} = \sum_{k=1}^{\beta_{l,j}} C_{A_l,j}^{(k)}.$$

We recall a fact from [B-S].

Proposition 3.2. For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{A_l}(J)$ is connected. Then, the degree $\deg(\Delta_{A_l,J})$ of the fundamental element is given by

$$\deg(\Delta_{A_l,J}) = \binom{\#(J) + 1}{2}.$$

From the equation (2.1), we easily show the following formula.

Proposition 3.3. For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{A_l}(J)$ has a decomposition $\Gamma_{A_l}(J) = \bigsqcup_{i=1}^k \Gamma_{A_l}(J_i)$. Then, the degree of the fundamental element $\Delta_{A_l,J}$ can be written as

$$\deg(\Delta_{A_l,J}) = \sum_{i=1}^k \deg(\Delta_{A_l,J_i}) = \sum_{i=1}^k \binom{\#(J_i) + 1}{2}.$$

Proposition 3.4. Let j and k be two positive integers with $j \leq l$ and $k \leq \beta_{l,j}$. For given positive integers τ_1, \dots, τ_k with $\sum_{i=1}^k \tau_i = j$, we define the set $S_{j,(\tau_1, \dots, \tau_k)}$ by

$$\left\{ J \subset I \mid \begin{array}{l} \#J = j, \Gamma_{A_l}(J) = \bigsqcup_{i=1}^k \Gamma_{A_l}(J_i) \text{ with } \min(J_1) < \dots < \min(J_k) \\ \text{s.t. } \#J_i = \tau_i (i = 1, \dots, k) \end{array} \right\}$$

Then, we have the following equation

$$\#S_{j,(\tau_1, \dots, \tau_k)} = \binom{l - j + 1}{k}.$$

We remark that the result does not depend on the choice of positive integers τ_1, \dots, τ_k . Hence, the number $\binom{l-j+1}{k}$ divides the number $C_{A_l, j}^{(k)}$. Then, we define the number $\tilde{C}_{A_l, j}^{(k)}$ by the equation

$$C_{A_l, j}^{(k)} = \tilde{C}_{A_l, j}^{(k)} \cdot \binom{l-j+1}{k}.$$

For two positive integers j, k with $k \leq j$, we put

$$T_{k, j} := \{(\tau_1, \dots, \tau_k) \in \mathbb{Z}_{\geq 0}^k \mid \sum_{i=1}^k \tau_i = j\}.$$

From the Proposition 3.3, we have

$$\tilde{C}_{A_l, j}^{(k)} = \sum_{(\tau_1, \dots, \tau_k) \in T_{k, j}} \sum_{i=1}^k \binom{\tau_i + 1}{2}.$$

Lemma 3.5. *Let j and k be two positive integers with $j \leq l$ and $k \leq \beta_{l, j}$. Then, the following equation $E_{j, k}$ holds.*

$$\tilde{C}_{A_l, j}^{(k)} = k \binom{j+1}{k+1}.$$

Proof. We will show the general equation $E_{j, k}$ by double induction. First, for $k = 1$, the subgraph $\Gamma_{A_l}(J)$ is connected. Hence, we easily compute $\deg(\Delta_{A_l, J}) = \binom{j+1}{2}$. Therefore, we say the equation $E_{j, 1}$ is true. Next, for induction hypothesis, we assume

(A) $E_{j, k}$ is true for $j = 1, \dots, r$ and arbitrary k ,
and

(B) $E_{r+1, k}$ is true for $1 \leq k \leq s-1$.

We will show the equation $E_{r+1, s}$. For a positive integer $i \in \{1, \dots, r-s+2\}$, we consider the set $\{(\tau_1, \dots, \tau_s) \in T_{s, j} \mid \tau_1 = i\}$. Then, we easily count the number $\#\{(\tau_1, \dots, \tau_s) \in T_{s, j} \mid \tau_1 = i\} = \binom{r-i}{s-2}$. By applying the induction hypothesis (A) and (B) to (τ_2, \dots, τ_s) , we have

$$\begin{aligned} \tilde{C}_{A_l, j}^{(s)} &= \sum_{i=1}^{r-s+2} \left\{ \binom{i+1}{2} \binom{r-i}{s-2} + \tilde{C}_{A_l, r+1-i}^{(s-1)} \right\} \\ &= \sum_{i=1}^{r-s+2} \binom{i+1}{2} \binom{r-i}{s-2} + (s-1) \sum_{i=1}^{r-s+2} \binom{r+2-i}{s} \\ &= \binom{r+2}{s+1} + (s-1) \binom{r+2}{s+1} = s \binom{r+2}{s+1}. \end{aligned}$$

This completes the proof. \square

Theorem 3.6. For positive integers l, j with $l \geq j$, the following equation holds:

$$C_{A_{l+1}, j+1} - C_{A_l, j} = (j+1) \binom{l+1}{j+1}.$$

Proof. First, we remark the following.

Proposition 3.7. *Theorem 3.6 implies Theorem 1.1 for the type A_l .*

Proof. It is easy to show $N'_{A_1}(1) = -1$. Hence, it suffices to show that $N'_{A_{l+1}}(1) + N'_{A_l}(1) = 0$ for any positive integer l .

$$\begin{aligned} & N'_{A_{l+1}}(1) + N'_{A_l}(1) \\ &= -C_{A_{l+1}, 1} + \sum_{j=1}^l (C_{A_{l+1}, j+1} - C_{A_l, j}) (-1)^{j+1} \\ &= -C_{A_{l+1}, 1} + \sum_{j=1}^l (-1)^{j+1} (j+1) \binom{l+1}{j+1} \\ &= \sum_{j=0}^l (-1)^{j+1} (j+1) \binom{l+1}{j+1} \\ &= -(l+1) \sum_{j=0}^l (-1)^j \binom{l}{j} \\ &= -\frac{d}{dx} (1+x)^{l+1} \Big|_{x=-1} = 0. \end{aligned}$$

□

To prove Theorem 3.6, we prepare a lemma.

Lemma 3.8. (1) For positive integers l, j with $j+1 \leq \lceil \frac{l+1}{2} \rceil$, the following equation holds:

$$\sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l-j+1}{k} = \binom{l+1}{j+1}.$$

(2) For positive integers l, j with $j+1 > \lceil \frac{l+1}{2} \rceil$, the following equation holds.

$$\sum_{k=1}^{l-j+1} \binom{j}{k-1} \binom{l-j+1}{k} = \binom{l+1}{j+1}.$$

Proof. (1) We rewrite the equation as follows

$$\sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l-j+1}{l-j+1-k} = \binom{l+1}{l-j}.$$

This follows from $(1+x)^{l+1} = (1+x)^j (1+x)^{l-j+1}$.

(2) In the same way, we obtain the result.

□

We consider two cases.

Case 1: $j + 1 \leq \lceil \frac{l+1}{2} \rceil$.

$$\begin{aligned}
& C_{A_{l+1}, j+1} - C_{A_l, j} \\
&= \sum_{k=1}^{j+1} C_{A_{l+1}, j+1}^{(k)} - \sum_{k=1}^j C_{A_l, j}^{(k)} \\
&= \sum_{k=1}^{j+1} k \binom{j+2}{k+1} \binom{l-j+1}{k} - \sum_{k=1}^j k \binom{j+1}{k+1} \binom{l-j+1}{k} \\
&= (j+1) \binom{l-j+1}{j+1} + \sum_{k=1}^j k \binom{j+1}{k} \binom{l-j+1}{k} \\
&= \sum_{k=1}^{j+1} k \binom{j+1}{k} \binom{l-j+1}{k} \\
&= (j+1) \sum_{k=1}^{j+1} \binom{j}{k-1} \binom{l-j+1}{k}.
\end{aligned}$$

Thanks to the Lemma 3.8 (1), we have

$$C_{A_{l+1}, j+1} - C_{A_l, j} = (j+1) \binom{l+1}{j+1}.$$

Case 2: $j + 1 > \lceil \frac{l+1}{2} \rceil$.

$$\begin{aligned}
& C_{A_{l+1}, j+1} - C_{A_l, j} \\
&= \sum_{k=1}^{l-j+1} C_{A_{l+1}, j+1}^{(k)} - \sum_{k=1}^{l-j+1} C_{A_l, j}^{(k)} \\
&= \sum_{k=1}^{l-j+1} k \binom{j+2}{k+1} \binom{l-j+1}{k} - \sum_{k=1}^{l-j+1} k \binom{j+1}{k+1} \binom{l-j+1}{k} \\
&= \sum_{k=1}^{l-j+1} k \binom{j+1}{k} \binom{l-j+1}{k} \\
&= (j+1) \sum_{k=1}^{l-j+1} \binom{j}{k-1} \binom{l-j+1}{k}.
\end{aligned}$$

Thanks to the Lemma 3.8 (2), we have

$$C_{A_{l+1}, j+1} - C_{A_l, j} = (j+1) \binom{l+1}{j+1}.$$

This completes the proof of Theorem 3.6. \square

4. Proof of the type B_l

Let l be a positive integer in $\mathbb{Z}_{\geq 2}$ and let $I = \{1, 2, \dots, l\}$. In this section, we will prove Theorem 1.1 for the type B_l . We recall a fact from [B-S].

Proposition 4.1. *For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{B_l}(J)$ is connected. Then, the degree $\deg(\Delta_{B_l, J})$ of the fundamental element is given by*

$$\deg(\Delta_{B_l, J}) = \begin{cases} \#(J)^2 & \text{if } J \supset \{1, 2\} \\ \binom{\#(J)+1}{2} & \text{if } J \not\supset \{1, 2\} \end{cases}$$

From the equation (2.1), if the full subgraph $\Gamma_{B_l}(J)$ for a subset $J \subset I$ has a decomposition $\Gamma_{B_l}(J) = \bigsqcup_{i=1}^k \Gamma_{B_l}(J_i)$ with $\min(J_1) < \dots < \min(J_k)$, then we can compute the degree $\deg(\Delta_{B_l, J})$ of the fundamental element. In the case of $J_1 \not\supset \{1, 2\}$, we compute

$$\deg(\Delta_{B_l, J}) = \sum_{i=1}^k \deg(\Delta_{B_l, J_i}) = \sum_{i=1}^k \binom{\#(J_i)+1}{2}. \quad (4.1)$$

Moreover, in the case of $J_1 \supset \{1, 2\}$, we compute

$$\deg(\Delta_{B_l, J}) = \sum_{i=1}^k \deg(\Delta_{B_l, J_i}) = \#(J_1)^2 + \sum_{i=2}^k \binom{\#(J_i)+1}{2}. \quad (4.2)$$

Theorem 4.2. *The following equation holds:*

$$N'_{B_l}(1) - N'_{A_l}(1) = (-1)^l(l-1).$$

Proof. We compute the difference between $N'_{B_l}(1)$ and $N'_{A_l}(1)$. From the equations (4.1) and (4.2), we only have to count the case when the set J_1 contains the index set $\{1, 2\}$. For a positive integer $u \in \{2, \dots, l-2\}$ and $X_l \in \{A_l, B_l\}$, we put

$$S_{X_l, u} := \left\{ J \subset I \mid \begin{array}{l} \Gamma_{X_l}(J) = \bigsqcup_{i=1}^k \Gamma_{X_l}(J_i) \text{ with } \min(J_1) < \dots < \min(J_k) \\ \text{s.t. } J_1 = \{1, \dots, u\} \end{array} \right\}$$

For each $u \in \{2, \dots, l-2\}$, the difference on $S_{X_l, u}$ is the following

$$\begin{aligned} & \sum_{J \in S_{B_l, u}} (-1)^{\#J} \deg(\Delta_{B_l, J}) - \sum_{J \in S_{A_l, u}} (-1)^{\#J} \deg(\Delta_{A_l, J}) \\ &= \sum_{J \in S_{B_l, u}} (-1)^{\#J} \left\{ \deg(\Delta_{B_l, J}) - \deg(\Delta_{A_l, J}) \right\} \\ &= \sum_{J \in S_{B_l, u}} (-1)^{\#J} \left\{ u^2 - \binom{u+1}{2} \right\} \end{aligned}$$

$$= \binom{u}{2} \sum_{J \in S_{B_l, u}} (-1)^{\#J} = 0.$$

Hence, we only have to count the cases $J_1 = \{1, \dots, l-1\}, \{1, \dots, l\}$

$$\begin{aligned} & N'_{B_l}(1) - N'_{A_l}(1) \\ &= (-1)^{l-1} \left\{ (l-1)^2 - \binom{l}{2} \right\} + (-1)^l \left\{ l^2 - \binom{l+1}{2} \right\} \\ &= (-1)^l (l-1). \end{aligned}$$

This completes the proof. \square

5. Proof of the type D_l

Let l be a positive integer in $\mathbb{Z}_{\geq 4}$ and let $I = \{1, 2, \dots, l\}$. In this section, we will prove Theorem 1.1 for the type D_l . We recall a fact from [B-S].

Proposition 5.1. *For a subset $J \subset I$, we suppose that the full subgraph $\Gamma_{D_l}(J)$ is connected. Then, the degree $\deg(\Delta_{D_l, J})$ of the fundamental element is given by*

$$\deg(\Delta_{D_l, J}) = \begin{cases} \binom{\#(J)+1}{2} & \text{if } J \not\supset \{l-1, l\} \\ \#(J)(\#(J)-1) & \text{if } J \supset \{l-3, l-2, l-1, l\} \\ 6 & \text{if } J = \{l-2, l-1, l\} \end{cases}$$

From the equation (2.1), if the full subgraph $\Gamma_{D_l}(J)$ for a subset $J \subset I$ has a decomposition $\Gamma_{D_l}(J) = \Gamma_{D_l}(J_1) \sqcup \dots \sqcup \Gamma_{D_l}(J_k)$ with $\min(J_1) < \dots < \min(J_k)$, then we can compute the degree $\deg(\Delta_{D_l, J})$ of the fundamental element.

Theorem 5.2. *The following equality holds:*

$$N'_{D_l}(1) = (-1)^l (l-2).$$

Proof. We will show the statement by induction on l . First, for $l = 4$, we easily compute $N'_{D_4}(1) = \sum_{J \subset I} (-1)^{\#J} \deg(\Delta_{D_4, J}) = 12$. Next, by applying the induction hypothesis, we will compute the difference $N'_{D_l}(1) - N'_{D_{l-1}}(1)$. For a positive integer $u \in \{1, \dots, l-2\}$, we put

$$S_{D_l, u} := \left\{ J \subset I \mid \begin{array}{l} \Gamma_{D_l}(J) = \bigsqcup_{i=1}^k \Gamma_{D_l}(J_i) \text{ with } \min(J_1) < \dots < \min(J_k) \\ \text{s.t. } J_1 = \{1, \dots, u\} \end{array} \right\}$$

$$I_u := I \setminus \{1, \dots, u, u+1\}.$$

For each $u \in \{1, \dots, l-5\}$, the difference on $S_{D_l, u}$ is the following

$$\sum_{J \in S_{D_l, u}} (-1)^{\#J} \deg(\Delta_{D_l, J})$$

$$\begin{aligned}
&= (-1)^u \sum_{K \subset I_u} (-1)^{\#K} \left\{ \deg(\Delta_{D_l, J_1}) + \deg(\Delta_{D_l, K}) \right\} \\
&= (-1)^u \deg(\Delta_{D_l, J_1}) \sum_{K \subset I_u} (-1)^{\#K} + (-1)^u \sum_{K \subset I_u} (-1)^{\#K} \deg(\Delta_{D_l, K}) \\
&= (-1)^u \sum_{K \subset I_u} (-1)^{\#K} \deg(\Delta_{D_l, K}) \\
&= (-1)^u N'_{D_{l-u-1}}(1).
\end{aligned}$$

From the induction hypothesis, this is equal to $(-1)^{l-1}(l-u-3)$. For $u = l-4$, the difference on $S_{D_l, l-4}$ is computed in a similar manner

$$\begin{aligned}
&\sum_{J \in S_{D_l, l-4}} (-1)^{\#J} \deg(\Delta_{D_l, J}) \\
&= (-1)^{l-4} N'_{A_3}(1).
\end{aligned}$$

For $u = l-3$, we easily compute

$$\sum_{J \in S_{D_l, l-3}} (-1)^{\#J} \deg(\Delta_{D_l, J}) = 0.$$

Therefore, we can compute the difference

$$\begin{aligned}
&N'_{D_l}(1) - N'_{D_{l-1}}(1) \\
&= \sum_{u=1}^{l-3} \sum_{J \in S_{D_l, u}} (-1)^{\#J} \deg(\Delta_{D_l, J}) + (-1)^{l-2} \binom{l-1}{2} \\
&= (-1)^{l-1} \left\{ (l-4) + (l-3) + \cdots + 1 \right\} + (-1)^{l-2} \binom{l-1}{2} \\
&= (-1)^l (2l-5).
\end{aligned}$$

From the induction hypothesis, we have

$$N'_{D_l}(1) = (-1)^l (l-2).$$

□

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